Displacement, Velocity, Acceleration

Definition.

- **Displacement** (s): directed distance from a fixed origin.
- **Velocity** (v): rate of change of displacement with respect to time.
- **Acceleration** (a): rate of change of velocity with respect to time.

These are *vector* quantities — they have direction, carried by their sign. The corresponding *scalar* quantities are distance and speed.

Fact (Units) — displacement m; velocity $\mathrm{m\,s^{-1}}$; acceleration $\mathrm{m\,s^{-2}}$.
An acceleration of $3\,\mathrm{m\,s^{-2}}$ means the velocity increases by $3\,\mathrm{m\,s^{-1}}$ every second.

Example

A particle moves 8 m to the right, then 3 m to the left, taking 5 s in total. Find

1. the distance travelled and the final displacement;
2. the average speed and the average velocity.

1. *Distance* 11 m; *displacement* +5 m.
2. *Average speed* $= \frac{11}{5} = 2.2\,\mathrm{m\,s^{-1}}$; *average velocity* $= \frac{5}{5} = 1\,\mathrm{m\,s^{-1}}$.

Example

A car travelling at $90\,\mathrm{km\,h^{-1}}$ brakes uniformly to rest in 8 s. Find its acceleration in $\mathrm{m\,s^{-2}}$.

$90\,\mathrm{km\,h^{-1}} = \frac{90000}{3600} = 25\,\mathrm{m\,s^{-1}}$

$a = \dfrac{0-25}{8} = -3.125\,\mathrm{m\,s^{-2}}$ *(negative: a deceleration).*

Textbook Exercises: SPS Course 4.9, Exercise 2 Q1–4

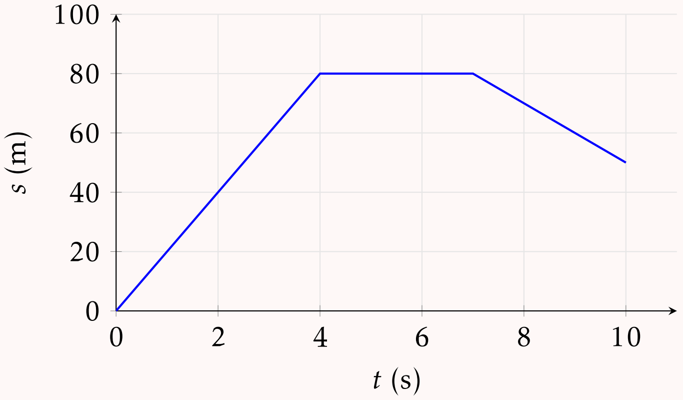
Displacement–Time Graphs

Fact — On a displacement–time graph:

- the gradient is the velocity (constant gradient = constant velocity);
- a horizontal section means the particle is stationary;
- negative gradient means the particle is moving back towards (or past) the origin.

Example

The graph shows the displacement of a particle (in m, s).



Find the velocity at $t = 3$, the velocity at $t = 9$, the average velocity over the 10 seconds, and the average speed.

At $t = 3$: $v = \frac{80}{4} = 20\,\mathrm{m\,s^{-1}}$. *At $t = 9$:* $v = \frac{50-80}{3} = -10\,\mathrm{m\,s^{-1}}$.
Average velocity $= \frac{50}{10} = 5\,\mathrm{m\,s^{-1}}$. *Average speed* $= \frac{80+30}{10} = 11\,\mathrm{m\,s^{-1}}$.

Example

Fred and Harold travel separately from London to Brighton, 80 km away. Fred sets off at noon at $30\,\mathrm{km\,h^{-1}}$; Harold sets off at 1 pm at $50\,\mathrm{km\,h^{-1}}$. Find exactly when and where Harold overtakes Fred.

With t in hours after noon: Fred $s = 30t$, Harold $s = 50(t-1)$.
$30t = 50(t-1) \implies t = 2.5$: *at 2.30 pm,* $s = 75\,\mathrm{km}$ *from London.*

Textbook Exercises: SPS Course 4.9, Exercise 2 Q5, 6, 8, 11

Velocity–Time Graphs

Fact — On a velocity–time graph:

- the gradient is the acceleration;
- the area between the graph and the t-axis is the displacement (areas below the axis count as negative).

Example

A car accelerates uniformly from rest to V m s^{-1} in 8 s, holds that speed for 12 s, then decelerates uniformly to rest in 5 s. The total distance travelled is 370 m. Sketch the velocity–time graph and find V.

Area (trapezium) $= \frac{1}{2}(8)(V) + 12V + \frac{1}{2}(5)(V) = 18.5V$
$18.5V = 370 \implies V = 20\,\text{m}\,\text{s}^{-1}$.

Example

A train accelerates uniformly from rest to $24\,\mathrm{m\,s^{-1}}$ in $60\,\mathrm{s}$, travels at this speed for 5 minutes, then decelerates uniformly to rest at $0.8\,\mathrm{m\,s^{-2}}$.

1. Sketch the velocity–time graph.
2. Find the total distance travelled.
3. Find the average speed for the whole journey.

Deceleration phase: $\frac{24}{0.8} = 30\,\mathrm{s}$.

2. $s = \frac{1}{2}(60)(24) + (300)(24) + \frac{1}{2}(30)(24) = 720 + 7200 + 360 = 8280\,\mathrm{m}$.

3. Average speed $= \frac{8280}{390} = 21.2\,\mathrm{m\,s^{-1}}$ *(3 s.f.).*

Example

A ball is thrown vertically upwards at $20\,\mathrm{m\,s^{-1}}$ and is in flight for $4\,\mathrm{s}$ (take the acceleration to be $-10\,\mathrm{m\,s^{-2}}$ throughout). Sketch the velocity–time graph, and interpret the two triangles it makes with the t-axis.

Straight line from $(0, 20)$ *to* $(4, -20)$, *crossing the axis at* $t = 2$ *(the highest point).*
Triangle above the axis: $\frac{1}{2}(2)(20) = 20\,\mathrm{m}$ *up. Triangle below:* $20\,\mathrm{m}$ *down.*
Total displacement 0*; total distance* $40\,\mathrm{m}$.

Textbook Exercises: SPS Course 4.9, Exercise 2 Q7, 9, 10, 12 and Revision Exercise 4.9

Curved Graphs

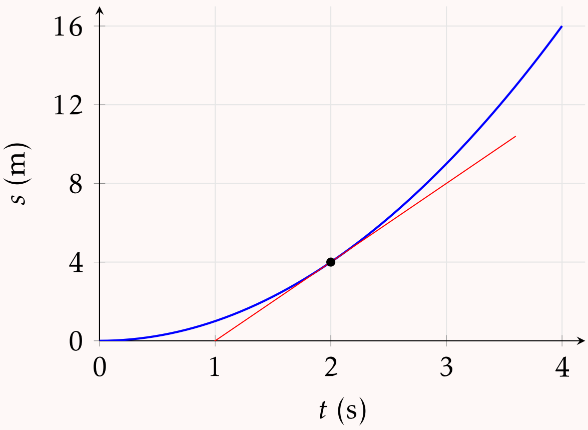
A curved displacement–time graph means the velocity is changing; a curved velocity–time graph means the acceleration is changing.

Fact — The gradient of a curve at a point is the gradient of the **tangent** at that point. Draw the tangent, form a right-angled triangle from it, and read off $\frac{\text{change in } y}{\text{change in } x}$ using the *axis scales* — not by counting squares.

Remark. The gradient is *not* the gradient of the line from the origin to the point.

Example

The graph shows $s = t^2$ for a falling particle, with the tangent drawn at $t = 2$. Find the velocity at $t = 2$.



Tangent passes through $(1, 0)$ *and* $(3, 8)$*:* $v = \frac{8-0}{3-1} = 4\,\text{m}\,\text{s}^{-1}$.

Estimating the Area Under a Curve

Fact — Split the region into vertical strips and treat each strip as a trapezium:

$$\text{area of strip} \approx \frac{h}{2}(v_{\text{left}} + v_{\text{right}})$$

where h is the strip width. Whether this over- or under-estimates depends on which way the curve bends — decide from a sketch.

Example

The velocity of a particle is recorded each second:

t (s)	0	1	2	3	4
v ($\mathrm{m\,s^{-1}}$)	0	1.8	3.2	4.2	4.8

1. Estimate the distance travelled in the 4 seconds using four trapezia.
2. The velocity is in fact increasing with decreasing acceleration throughout. Is your estimate an over-estimate or an under-estimate?

1. $s \approx \frac{1}{2}(0+1.8)+\frac{1}{2}(1.8+3.2)+\frac{1}{2}(3.2+4.2)+\frac{1}{2}(4.2+4.8) = 11.6\,\mathrm{m}$

2. The curve is concave (bends downwards), so each chord lies below the curve: under-estimate.

Exercise. Sketch $y = x^3 - x$. Using tangents at a few points (and symmetry), sketch the graph of its *gradient* against x. What type of function does the gradient graph appear to be?

Textbook Exercises: SPS Course 4.9, Exercise 1 and Exercise 2 Q13–16, 18